ARGOS - Arithmetische Geometrie Oberseminar

# The Arithmetic Fundamental Lemma

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The aim of this semester's seminar is to work through Wei Zhang's proof of the Arithmetic Fundamental Lemma (AFL), see [7].

**Theorem** (Arithmetic Fundamental Lemma, Zhang 2019). *Let $g \in U(V)$ be regular semi-simple, matching the element $\gamma \in S_n(\mathbb{Q}_p)$. Then, up to an explicit sign, there is an equality*

$$\pm\partial O(\gamma, 1_{S_n(\mathbb{Z}_p)}) = \mathrm{Int}(g)\log(p).$$

Here, $U(V)$ is the unitary group of an $n$-dimensional non-split hermitian space for the unramified extension $\mathbb{Q}_{p^2}/\mathbb{Q}_p$. It acts on a certain Rapoport-Zink space and $\mathrm{Int}(g)$ is the intersection number of a certain cycle with its $g$-translate. The left hand side is a certain analytic-combinatorial quantity defined as the derivative at $s = 0$ of an orbital integral over a symmetric space $S_n \subset \mathrm{Res}_{\mathbb{Q}_{p^2}/\mathbb{Q}_p} GL_n$ with respect to a standard test function $1_{S_n(\mathbb{Z}_p)}$. The elements $g$ and $\gamma$ are related by a comparison of invariants with respect to certain group actions, which implies e.g. that they have the same characteristic polynomial.

Zhang's proof of the AFL is of a global nature. On the analytic side, he defines certain distributions with values in holomorphic modular forms. This is where the Weil representation comes into play. Similarly, on the intersection theoretic side, he defines holomorphic modular forms by intersecting the generating series from [1] with CM-cycles. The choices can be made such that the AFL identity in question occurs as the comparison of a certain Fourier coefficient.

Local intersection numbers (resp. derived orbital integrals) from finitely many prime numbers $p$ contribute to a fixed Fourier coefficient. At almost all primes, the order used to define the CM-cycle is maximal and the two contributions can be directly compared, cf. [2]. For a bad prime $p$, at least its contributions to Fourier coefficients in degree prime to $p$ can be compared by an inductive argument. This already implies the equality of the two modular forms in question and hence the desired AFL identity.

## Talks

Citations refer to [7] if not stated otherwise. You may assume that $F_0 = \mathbb{Q}$ resp. $F_0 = \mathbb{Q}_p$ to simplify the exposition in your talk.

### Talk 1: Matching and orbital integrals

Introduce the group-theoretic set up from Section 2.1 and the matching of orbits from Section 2.2. A proof of the bijectivity of the matching relation (for the case of groups) can be found in [5, Lemma 2.3]. Define the relevant orbital integrals and the transfer of test functions as in Section 2.3. Note that by [6, Theorem 2.6], a smooth transfer always exists.

### Talk 2: The Fundamental Lemma

State the Fundamental Lemma from Section 2.4. Reinterpret the occurring orbital integrals as certain lattice counts as in [3, Section 7] (group version) or [4, Section 2] (Lie algebra version). Adapt these results to the semi-Lie algebra version, cf. [2, Section 9]. Show the vanishing part of the Fundamental Lemma as in [3, Corollary 7.3]. Prove the case of complex multiplication Proposition 2.6 and, if time permits, the induction statement Proposition 2.7.

**Talk 3: The Arithmetic Fundamental Lemma**

Follow Section 3.1 to define both the group and semi-Lie algebra version of the AFL. For the semi-Lie algebra version, introduce the necessary background on $K$-groups from Appendix B. In order to show that the intersection number $\mathrm{Int}(g)$ (resp. $\mathrm{Int}(g,u)$) is well defined, prove that the intersection $\Delta \cap g\Delta$ (resp. $\mathcal{N}_n^g \cap \mathcal{Z}(u)$) is a projective scheme over $\mathrm{Spf}\, \mathcal{O}_{\breve{F}}$.

**Talk 4: Special cases of the AFL**

Prove the AFL in the case $n = 2$, see [5, Section 2.3]. Use [2, Theorems 10.1 and 10.5] to deduce the case of complex multiplication, see Proposition 3.9. Formulate both the equivalence of group and semi-Lie algebra version and the induction scheme, see Proposition 4.12. Sketch its proof.

**Talk 5: Local constancy of intersection numbers**

Present the results of Section 5, concerning the local constancy of the intersection numbers $\mathrm{Int}(g)$ and $\mathrm{Int}(g,u)$. For this, you will also have to prove Lemma 3.8 on the horizontal/vertical decomposition of the derived local CM-cycle from Talk 3. Also state Lemma 13.9.

**Talk 6: Shimura varieties and fat big CM-cycles**

Define the unitary Shimura varieties and the integral models that play a role for the globalization of the AFL setting, see Section 6. Introduce the fat big CM-cycles on these varieties as in Section 7.3 and endow them with a derived structure by expressing them as fixed points of a Hecke correspondence, see Sections 7.5 and 7.7.

**Talk 7: Kudla-Rapoport divisors**

Define the global KR-divisors as in Section 7.1. Endow these with Green's functions to define them as elements of the (reduced) Arakelov Chow group, see Section 8.2. Give Theorem 8.2 on the comparison of the two possible choices of Green's functions. Define the generating series with KR-divisors as coefficients and state the results on their modularity (Theorems 8.1 and 8.3).

**Talk 8: Intersecting CM-cycles and KR-divisors**

Follow Sections 9 and 10 to express the intersection of a fat big CM-cycle and the generating series from the previous talk in terms of local intersection numbers. The main results are Theorems 9.3 and 10.1 as well as Corollary 10.2. To prove Theorem 9.3, you also have to introduce the $p$-adic uniformization of the basic locus of the Shimura variety and of the cycles in question, see Sections 7.2 and 7.4.

**Talk 9: Weil representation and RTF with Gaussian test functions**

Use the Weil representation to define the global distribution $\mathbb{J}(h, \Phi', s)$ from Section 11, which is the analog on the analytic side of the generating series from the previous talk. Define the notion of (partial) Gaussian test functions and explain the regularization occurring in the definition of $\mathbb{J}$, see Section 12.6. State the decomposition at $s = 0$, see Section 12.7.

**Talk 10: Proof of the AFL**

Present the constant term computation "AFL at archimedean place" from Section 14.1. Finish the proof of the AFL, see Sections 14 and 15.

## References


[1] Bruinier, J., Howard, B., Kudla, S., Rapoport, M., Yang, T.: *Modularity of generating series of divisors on unitary Shimura varieties*, Preprint, arXiv:1702.07812 (2017).

[2] Mihatsch, A.: *Relative unitary RZ-spaces and the arithmetic fundamental lemma*, to appear in J. Inst. Math. Jussieu, arXiv:1611.06520 (2016).

[3] Rapoport, M., Terstiege, U., Zhang, W.: *On the Arithmetic Fundamental Lemma in the minuscule case*, Compos. Math. **149**, 1631-1666 (2013).

[4] Yun, Zhiwei: *The fundamental Lemma of Jacquet and Rallis*, Duke Math. J. **156**, 167–228 (2011).

[5] Zhang, Wei: *On arithmetic fundamental lemmas*, Invent. Math. **188**, 197–252 (2012).

[6] Zhang, Wei: *Fourier transform and the global Gan-Gross-Prasad conjecture for unitary groups*, Ann. of Math. **180**, 971-1049 (2014).

[7] Zhang, Wei: *Weil representation and arithmetic fundamental lemma*, Preprint (2019).